IMAGE RESTORATION ALGORITHMS BASED ON THE BISPECTRUM

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ABSTRACT

In this paper we propose two algorithms for the restoration of images based on the bispectrum. The bispectrum of a signal is the Fourier transform of its triple correlation. While second-order statistics (e.g., correlation function, power spectrum, etc.) do not provide any information about the phase of the signal, third-order statistics (e.g., triple correlation, bispectrum, etc.) allow the recovery of the phase of the signal. We propose two algorithms for estimating the magnitude and the phase of the image, where the ambiguity due to the use of the principal value of the phase component is taken into account. Image lines are used in our experiments to test the effectiveness of the proposed algorithms.

1. INTRODUCTION

In the Fourier representation of images[1], spectral magnitude and phase play different roles [1]. Many of the important features of an image are preserved if only the phase, but not if only the magnitude, is retained [1, 2]. The importance of the phase seems to be overlooked in many existing image restoration algorithms. More specifically, if we consider the constrained least-squares (CLS) filter [3, 4] and the Wiener filter [3] as two representative examples, both these filters have the Fourier phase of an inverse filter, since they minimize the magnitude of the restoration error, disregarding the phase. As a result, the restored image is inaccurate, especially at low signal-to-noise ratios (SNR), since the error in the phase of the restored image increases as the amount of noise increases.

It is natural to expect that better restoration results can be achieved by using an accurate estimate of the phase of the original image in the restoration process. A considerable amount

[1]In this paper we use the words signal and image interchangeably, since the results apply to signals of any dimensionality. For convenience, the notation refers to one-dimensional signals.

of work has appeared recently in the literature (see, for example, [5]-[8]) on the estimation of the phase using higher order spectra (e.g., bispectrum). Since the bispectrum is insensitive to additive Gaussian noise and preserves the phase information of the original image, it proves to be very useful in solving the restoration problem. In image restoration, as well as in other applications, the principal value of the phase of the original image, not necessarily the continuous phase, is adequate. In other words, a phase ambiguity factor of $2\pi p$, where p is an integer, is allowable. Therefore, in this paper, our main focus is the estimation of the principal value of the phase of the original image from the principal value of the bispectral phase of the observed image.

Two image restoration algorithms are developed based on the bispectrum. The magnitude of the original image is computed in a quite straightforward way from the magnitude of the bispectrum. A major problem exists, however, in estimating the phase of the image from the phase of its bispectrum; that is, if the phase of the bispectrum is not within the principal value range, there is a $2\pi k$ (k is an integer) phase ambiguity, which results in erroneous estimates of the (principal value) of the phase of the original signal. Some of the existing algorithms in the literature [5] ignore this $2\pi k$ phase ambiguity problem. Only very recent work has been reported in the literature identifying and providing solutions to this problem [6, 7, 8]. This paper proposes two ways to overcome the phase ambiguity problem without phase unwrapping.

This paper is organized as follows: In Sec. 2, an expression for the bispectrum of a noisy blurred signal and some of its properties are shown. In Sec. 3, we propose two algorithms to avoid the phase ambiguity problem. In Sec. 4, experimental results with one-dimensional signals are shown and Sec. 5 concludes the paper.

2. BISPECTRUM OF A NOISY BLURRED SIGNAL

Modeling the observed signal $y(n)$ as the output of a linear space invariant system corrupted by additive noise, the degradation equation takes the form

$$y(n) = d(n) * x(n) + v(n) \ , \tag{1}$$

where $*$ denotes convolution, $d(n)$ is the deterministic point spread function (psf) of the blurring system, $x(n)$ is the deterministic original signal and $v(n)$ is the stochastic additive noise, which is assumed to be Gaussian with zero-mean. The bispectrum of $y(n)$ is equal to

$$B_y(k_1, k_2) = E[B_{ys}(k_1, k_2)], \tag{2}$$

where B_{ys} is a sample bispectrum, which means the bispectrum of one deterministic observation, $E[\cdot]$ denotes expectation and

$$B_{ys}(k_1, k_2) \ = \ Z(k_1)Z(k_2)Z^*(k_1 + k_2) + Z(k_1)Z(k_2)V^*(k_1 + k_2)$$

$$+Z(k_1)V(k_2)Z^*(k_1+k_2)+V(k_1)Z(k_2)Z^*(k_1+k_2)$$
$$+Z(k_1)V(k_2)V^*(k_1+k_2)+V(k_1)Z(k_2)V^*(k_1+k_2)$$
$$+V(k_1)V(k_2)Z^*(k_1+k_2)+V(k_1)V(k_2)V^*(k_1+k_2), \quad (3)$$

where $Z(k) = D(k)X(k)$ and $D(k)$, $X(k)$ and $V(k)$ are the DFT sequences of $d(n)$, $x(n)$ and $v(n)$, respectively. Since $E[V(k)] = 0$ and $E[V(k_1)V(k_2)V^*(k_1+k_2)] = 0$, Eq. (2) takes the form [8]

$$B_y(k_1,k_2) = Z(k_1)Z(k_2)Z^*(k_1+k_2) + NX(0)[S_{vv}(k_1)\delta(k_1+k_2) + S_{vv}(k_2)\delta(k_1) + S_{vv}(k_1)\delta(k_2)], \quad (4)$$

where $S_{vv}(k)$ is the discrete power spectrum of $v(n)$ and $\delta(m)$ represents the discrete unit sample function, that is, $\delta(m) = 1$, for $m = 0$ and $\delta(m) = 0$ for $m \neq 0$. If the mean of the observation is equal to zero or the three axes in the bispectral plane ($k_1 = 0$, $k_2 = 0$ and $k_1 = -k_2$) are not used, then

$$\begin{aligned} B_y(k_1,k_2) &= Z(k_1)Z(k_2)Z^*(k_1+k_2) \\ &= D(k_1)D(k_2)D^*(k_1+k_2)X(k_1)X(k_2)X^*(k_1+k_2), \end{aligned} \quad (5)$$

or

$$B_{\tilde{y}}(k_1,k_2) = X(k_1)X(k_2)X^*(k_1+k_2), \quad (6)$$

where

$$B_{\tilde{y}}(k_1,k_2) = \frac{B_y(k_1,k_2)}{D(k_1)D(k_2)D^*(k_1+k_2)} \quad for \quad D(k_1), \quad D(k_2) \quad and \quad D^*(k_1,k_2) \neq 0, \quad (7)$$

and

$$B_{\tilde{y}}(k_1,k_2) = 0 \quad for \quad D(k_1) = 0 \quad or \quad D(k_2) = 0 \quad or \quad D^*(k_1,k_2) = 0. \quad (8)$$

According to Eq. (6), the phase and the log-magnitude relationships between the bispectrum and the signal are given by

$$\psi(k_1,k_2) = \phi(k_1) + \phi(k_2) - \phi(k_1+k_2), \quad (9)$$

and

$$T(k_1,k_2) = S(k_1) + S(k_2) + S(k_1+k_2) \quad (10)$$

where $\psi(k_1,k_2)$ and $\phi(k)$ represent the phase of $B_{\tilde{y}}(k_1,k_2)$ and $X(k)$, respectively, $T(k_1,k_2)$ and $S(k)$ represent the log-magnitude of $B_{\tilde{y}}(k_1,k_2)$ and $X(k)$, respectively. The restoration problem clearly consists of the estimation of $\phi(k)$ and $S(k)$ from knowledge of $\psi(k_1,k_2)$ and $T(k_1,k_2)$, respectively. While the estimation of $S(k)$ is a quite straightforward problem, the estimation of $\phi(k)$ is complicated since only the principal value $\psi_p(k_1,k_2)$ of $\psi(k_1,k_2)$ is available, where $-\pi < \psi_p(k_1,k_2) \leq \pi$. Therefore, if the bispectral phase is not restricted in the principal value range, Eq. (9) should be modified to

$$\psi_p(k_1,k_2) = \phi(k_1) + \phi(k_2) - \phi(k_1+k_2) + 2\pi p(k_1,k_2), \quad (11)$$

in order to obtain exact values, where $p(k_1, k_2)$ is an integer. The factor $p(k_1, k_2)$ is causing the phase ambiguity problem. A number of algorithms which do not take it into account, result in erroneous estimates of $\phi(k)$, as is explained in Ref. [8] for Brillinger's and Matsuoka and Ulrych's algorithms.

3. PROPOSED RESTORATION ALGORITHMS

In this section, two algorithms are proposed which do not suffer from the phase ambiguity problem in estimating the phase of the original signal.

3.1. Algorithm 1.

In order to reduce the phase estimation errors due to noise, averaging of all the available estimates is necessary. However, the order of averaging is critical, since it can cause the phase ambiguity problem. Therefore, with the proposed algorithm we obtain the principal value of the various estimates of the phase along various paths and then average these values.

The proposed algorithm is as follows. We choose all paths in the region of the bispectral domain shown in Fig. 1. Let the estimated principal phase along the l-th path be denoted by $\phi_p^l(k)$. For each path an initial value $\phi_{p,init}^l$ is required. Assuming that the initial point along the first path $\phi_{p,init}^1$ is known, the initial points for the rest of the paths are defined by

$$\phi_{p,init}^l = \phi_p^l(l) = \frac{1}{l-1}\sum_{j=1}^{l-1}\phi_p^j(l), \quad l \geq 2. \tag{12}$$

This means that the initial value at each path is obtained by averaging the estimated phase values at the previous paths. This reduces the sensitivity of the initial estimate to the remaining noise in the bispectral domain. According to Fig. 1, the estimated phase along each path is written as

$$\phi_p^l(k+l) = PV[\phi_p^l(l) + \phi_p^l(k) - \psi_p(k,l)], \quad k = l, \ldots, \frac{N}{4},$$
$$with\ \ initial\ \ value\ \ \phi_{p,init}^l = \phi_p^l(l) = \frac{1}{l-1}\sum_{j=1}^{l}\phi_p^j(l). \tag{13}$$

where PV denotes the principal value operator. Since all estimated principal value phases have no error due to $2\pi k$ factors, averaging them will not cause any phase ambiguity problem. The final estimated phase is written as

$$\phi_p(n) = \frac{2}{n}\sum_{l=1}^{n/2}\phi_p^l(n), \quad n = 2,..,\frac{N}{2}, \quad for\ \ n\ \ even. \tag{14}$$

$$\phi_p(n) = \frac{2}{n-1}\sum_{l=1}^{(n-1)/2}\phi_p^l(n), \quad n = 3,..,\frac{N}{2}-1, \quad for\ \ n\ \ odd. \tag{15}$$

Recursive estimation of the log-magnitude of the signal is similar to the above recursion except that there is no principal value issue. The procedure is as follows :

$$S(n) = \frac{2}{n} \sum_{i=1}^{n/2} [-S(i) - S(n-i) + T(i, n-i)], \quad n = 2, .., \frac{N}{2}, \quad for \ \ n \ \ even, \tag{16}$$

$$S(n) = \frac{2}{n-1} \sum_{i=1}^{(n-1)/2} [-S(i) - S(n-i) + T(i, n-i)], \quad n = 3, .., \frac{N}{2} - 1, \quad for \ \ n \ \ odd. \tag{17}$$

3.2. Algorithm 2.

According to this algorithm Eq. (11) is first rewritten as

$$\Psi_p = (\ A \mid B \ ) \cdot \begin{pmatrix} \Phi \\ -- \\ 2\pi K \end{pmatrix}, \tag{18}$$

where Ψ_p and Φ are vectors formed by stacking $\psi_p(i, j)$ and $\phi(i)$ respectively, K is an integer vector accounting for the unwrapping of the phase, A is a constant matrix described by [5]

$$A = \begin{pmatrix} 2 & -1 & . & . & . \\ 1 & 1 & -1 & . & . \\ 1 & 0 & 1 & -1 & . \\ . & . & . & . & . \end{pmatrix} \tag{19}$$

and B is a constant matrix which presents the position of a discontinuity of the bispectral phase. It is determined by a thresholding operation denoting the magnitude of a principal jump. An example of this matrix is given by

$$B = \begin{pmatrix} 1 & 0 & . & . & . \\ 1 & 0 & . & . & . \\ 0 & 1 & 0 & . & . \\ 0 & 1 & 0 & . & . \\ 0 & 1 & 0 & . & . \\ 0 & 0 & 1 & 0 & . \\ . & . & . & . & . \end{pmatrix} \tag{20}$$

In the above matrix, the row at which a new column of "1s" starts indicates the position where a jump of an integer multiple of 2π in the phase of the bispectrum occured. A least squares solution of Eq. (18) is iteratively sought, that is,

$$Q(\Phi, K) = ||\Psi_p - A\Phi - 2\pi BK||^2 \tag{21}$$

is minimized with respect to Φ and K. This minimization task is separated since $Q(\Phi, K)$ is quadratic. That is, two iterations are used for obtaining K and Φ, respectively. The first iteration with respect to Φ starts with an initial condition of K_{init} and defining $\Psi_1 = \Psi_p - 2\pi BK$, the iteration takes the form

$$\Phi(t+1) = \Phi(t) + \beta[A^T\Psi_1 - A^T A\Phi(t)], \tag{22}$$

where t is an iteration index. After convergence, the second iteration solving for K runs with the Φ_{init} obtained in the previous step. Defining $\Psi_2 = \Psi_p - A\Phi$, the second iteration takes the form

$$K_i(t+1) = K_i(t) + d_i(u_i), \tag{23}$$

where

$$d_i(u_i) = \left\{ \begin{array}{ll} -1 & u_i < -\theta_i \\ 0 & -\theta_i \leq u_i < \theta_i \\ 1 & u_i > \theta_i \end{array} \right\},$$

$$u_i = \{B^T\Psi_2 - 2\pi B^T BK(t)\}_i,$$

$\theta_i = \frac{1}{2}b_{ii}$, with b_{ii} the diagonal element of $B^T B$ and the subscript i denotes the i-th component of a vector. The vector K obtained at convergence is used as an initial condition to rerun iteration (22). The convergence of this algorithm is shown for both iterations separately. For iteration (22), the difference between two successive iteration steps is equal to

$$\Phi(t+1) - \Phi(t) = (I - \beta A^T A)(\Phi(t) - \Phi(t-1)). \tag{24}$$

By the triangular inequality, we get

$$\|\Phi(t+1) - \Phi(t)\| \leq \|I - \beta A^T A\| \cdot \|\Phi(t) - \Phi(t-1)\|. \tag{25}$$

Therefore, if $\|I - \beta A^T A\| < 1$, the iteration converges to a unique point. This inequality results in

$$\max_i |\lambda_i(I - \beta A^T A)| < 1, \tag{26}$$

which can be described as

$$0 < \beta < 2 \cdot \|A\|^{-2}, \tag{27}$$

where $\lambda_i(A)$ denotes the i-th eigenvalue of the matrix A. The convergence analysis of the second iteration is in Ref. [9], where it is shown that the algorithm converges to a local minimum. The log-magnitude estimation is performed by using an iteration similar to (22), that is

$$S(t+1) = S(t) + \gamma[C^T T - C^T CS(t)], \tag{28}$$

where γ is the gain parameter and the matrix C is the same as the matrix A of (19) with the -1 components of A is replaced by +1.

4. EXPERIMENTAL RESULTS

Experimental results with 1-D signals are presented in this section. Consider a signal of 32 samples which represents an arbitrary line of an image with no noise. Its Fourier phase wrapped to the principal value range and the Fourier magnitude are respectively shown in Figs. 2a and 2b. Algorithms 1 and 2 are applied to estimate the phase and the log-magnitude of the signal. The results are shown in Fig 3a- 3d. The estimated phase and magnitude with each of the proposed algorithms are almost identical to the correct phase and magnitude in Figs. 2a and 2b. In all previous experiments noiseless data were used. We now present results showing the effect of blur and noise on the restored signal. An ensemble of 32 noisy blurred samples of an image line of length 128 samples is used as the test signal. The blur is due to motion over seven samples with $SNR = 10dB$. The criteria $\|\Phi(t+1) - \Phi(t)\|^2/\|\Phi(t)\|^2 \leq 10^{-6}$, $\|K(t+1) - K(t)\|^2 = 0$ and $\|S(t+1) - S(t)\|^2/\|S(t)\|^2 \leq 10^{-6}$ were used for terminating respectively iterations (22), (23) and (28). The performance of the restoration was evaluated by measuring the improvement in signal to noise ratio denoted by Δ_{SNR} and defined by

$$\Delta_{SNR} = 10 \log_{10} \frac{\|y_1 - x\|^2}{\|\hat{x} - x\|^2}.$$

where y_1 is the first realization of an observed ensemble and $\hat{x}$ is the restored signal. The original image line and a realization of the noisy blurred line ($SNR = 10dB$) are shown in Figs. 4a and 4b, respectively. The restored signals by Algorithms 1 and 2 are shown respectively in Fig. 4c ($\Delta_{SNR} = 3.7dB$) and 4d ($\Delta_{SNR} = 3.1dB$).

5. CONCLUSION

In this paper restoration algorithms have been proposed which estimate the magnitude and the phase of a signal from the bispectrum of a noisy-blurred version of it. Some of the algorithms reported in the literature for phase estimation using the bispectrum provide erroneous results when the phase of the signal and its bispectrum are not in the principal value range. Two algorithms were proposed which overcome such a limitation. The key idea in the first proposed algorithm is to obtain the principal value of the phase components and then perform an averaging operation, instead of the other way around, while in the second proposed algorithm is to include the principal jumps of the bispectral phase into a least-square method. As shown by the experimental results, the proposed algorithms recover the phase of the original signal almost errorlessly in noiseless cases. On the other hand, at low SNRs, although with some errors, the restored signals are still very satisfactory.

The error in the estimation of the phase at low SNRs is primarily due to the fact that although in theory the bispectrum of a Gaussian zero mean noise is zero, this is not the case in the experiments, since the numbers of realizations is finite. This error is larger at high frequencies.

This is due to the fact that the real and imaginary part (and therefore the magnitude) of the bispectrum are very small at high frequencies. Therefore, the estimates of the phase are susceptible to noise. The problem of filtering the noise in the bispectral domain, therefore obtaining more accurate estimates of the phase of a signal when only a small number of realizations are available, is currently under investigation.

6. ACKNOWLEDGMENT

This work was supported in part by a grant from Siemens.

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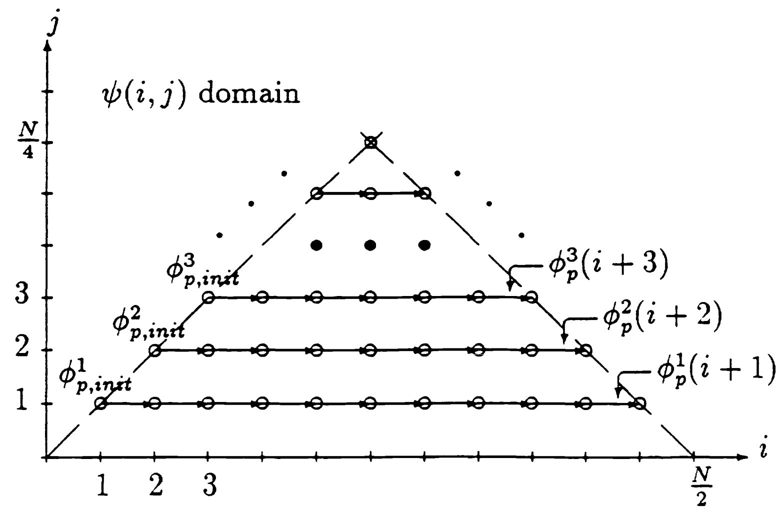

Fig. 1. Recursion path of Algorithm 1.

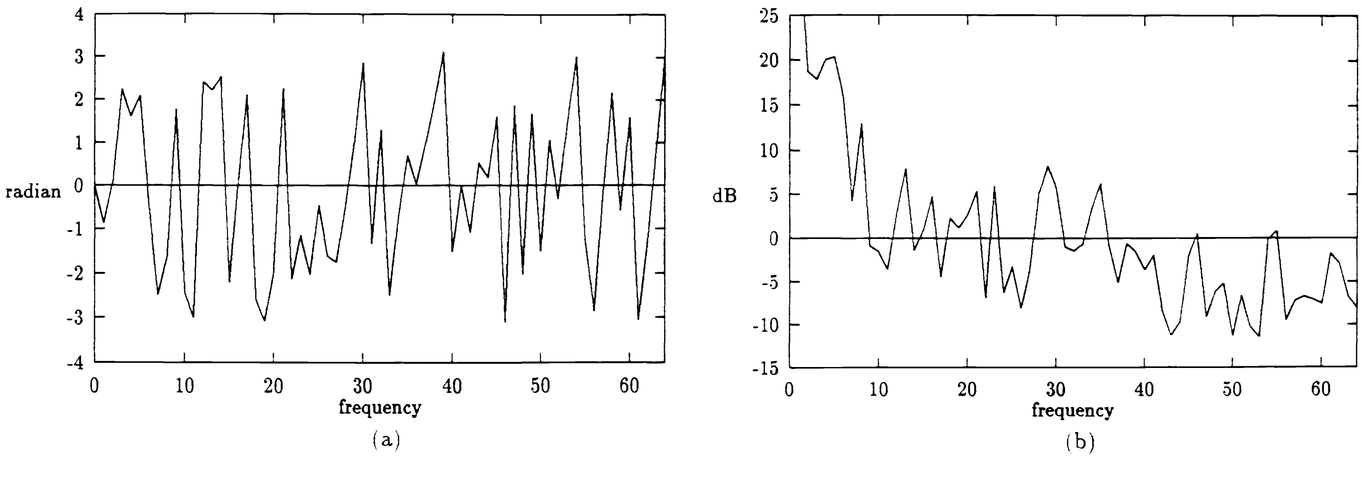

Fig. 2. (a) Phase of a 128-sample image line. (b) Spectral magnitude of a 128-sample image line.

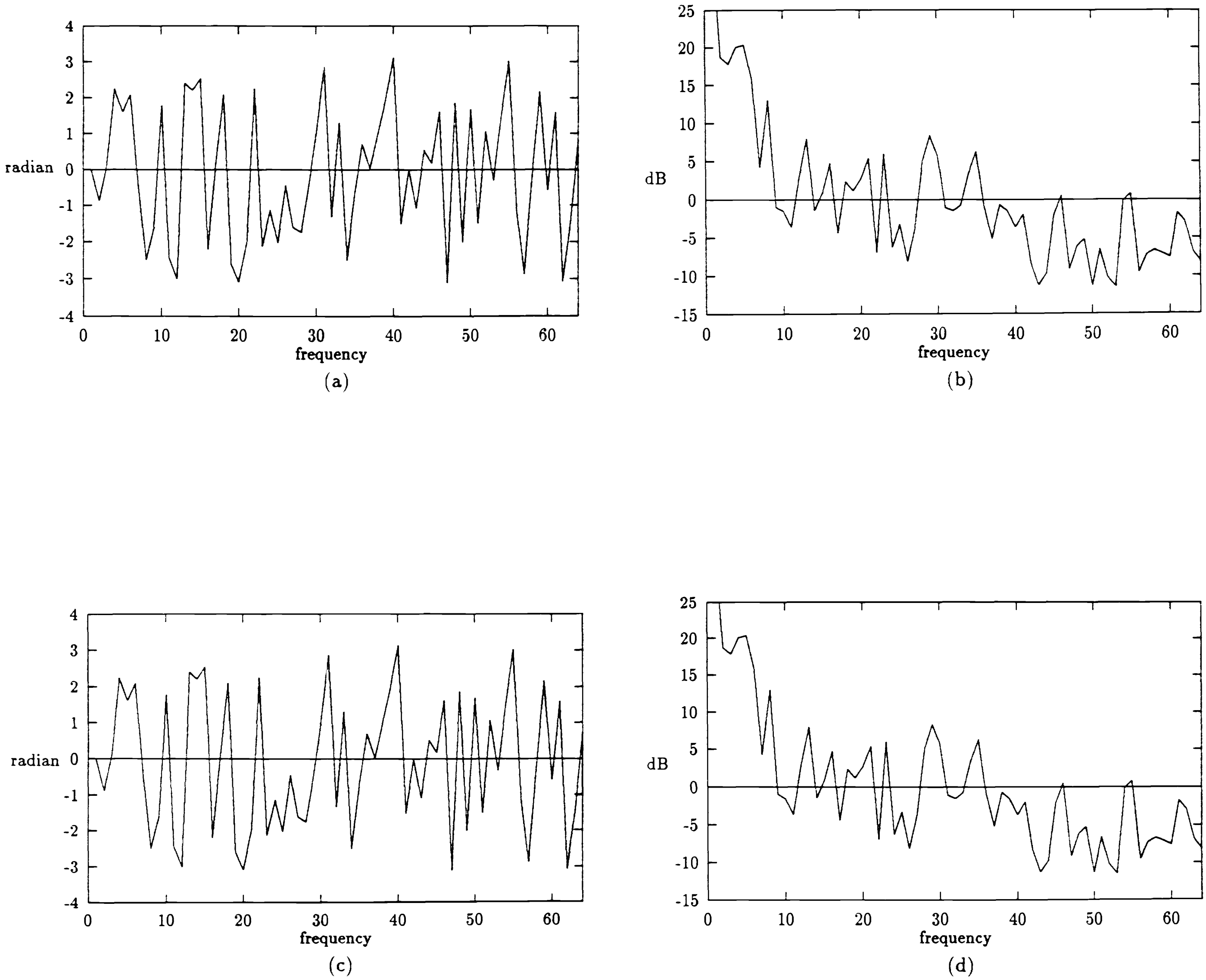

Fig. 3. Estimated phase and log-magnitude using: (a), (b) Algorithm 1, (c), (d) Algorithm 2. No noise.

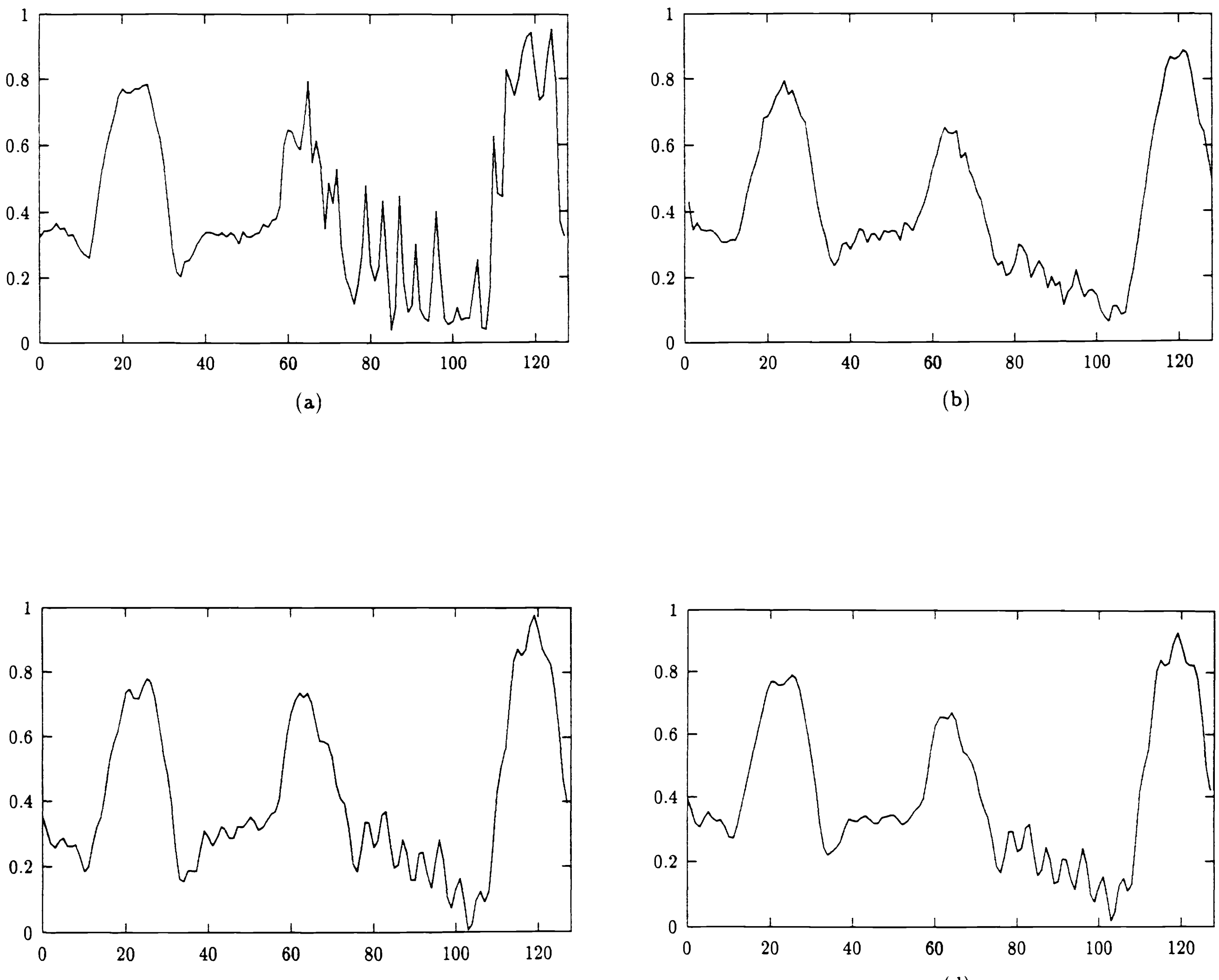

Fig. 4. (a) Original image line of 128 aamples. (b) Noisy blurred image line; seven-sample motion blur, SNR = 10 dB. Restored image line by (c) Algorithm 1, (d) Algorithm 2.